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Ivangoe B Pestov

Joint Institute for Nuclear Research, Bogoliubov Laboratory of Theoretical Physics, 141980, Dubna, Russia

E-mail: pestov@thsun1.jinr.ru

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Abstract. The concept of a wave field is introduced to represent oriented media. The wave field is a tensor field of second rank, and directors are its eigenvectors. This exhibition of directors defines a natural gauge group inherent in continua and allows one to derive from a variational principle general relativistic and gauge-invariant equations for the wave field in question. Thus, the gauge-theoretical approach to a continuum with internal degrees of freedom gives rise to an unambiguous and minimally coupled theory.

1. Introduction

In generalized continuum mechanics dealing with oriented media the fundamental structure is a set of vectors adjoined to each point of a medium. These vectors which can deform independently of the displacements of points of the media are called directors. They define a microstructure of the continuum. In the theory of spinning fluids, the name 'tetrads' is also used. At present, this part of mechanical science is fairly well accepted and is widely applied for the description of media possessing internal degrees of freedom.

There exists an extended literature on continua with internal degrees of freedom, see, for example, Cosserat and Cosserat [1], Weyssenhoff and Raabe [2], Maugin and Eringen [3], Halbwachs [4], Minkevich and Karakura [5], Berman [6], Moffatt [7], Holz [8] and references therein. The gauge-theoretical point of view is stressed by Kleinert [9]. In the theory of elastic continua with defects the gauge approach was successfully used by Osipov [10]. The theory of spinning fluids in generalized spacetime manifolds was developed by Ray and Smally [11] as an extension of the theory of spinning fluids in special relativity. The Cosserat brothers were the first to introduce the notion of a 3-tuple of unit rigid directors and laid down the mathematical foundations of the theory now known as the Cosserat continuum.

The goal of the present paper is to formulate the most general gauge-theoretical approach to the theory of oriented media. To do this, the notion of a wave field is introduced to represent generalized continua. The wave field is characterized so as to define directors and other properties of the system in question, in particular, the form of interaction of this matter with the gravitational field. The wave field allows one to derive a minimally coupled and simple theory from first principles which has a fundamental meaning in contemporary physics.

2. Formulation of the problem

The key idea of the present consideration is that directors are defined as solutions of the eigenvalue problem of the form

$$\Psi^i_j h^j = \lambda h^i \tag{1}$$

where the matrix Ψ^i_j, is a tensor field of the type (1, 1) and, at the same time, the wave field of oriented media. The internal state of media is then characterized by the number of linear independent eigenvectors (directors) h^i which define the type of microstructure of continua. Now, the problem is to derive a wave equation for the field Ψ^i_j from first principles.

We remark that equations (1) are invariant under the transformations

$$\bar{\Psi}^i_j = S^i_k \Psi^k_l T^l_j \tag{2}$$

$$\bar{h}^i = S^i_j h^j \tag{3}$$

where S^i_j are components of the tensor field S of type (1, 1) that satisfy the condition $\det(S^i_j) \neq 0$. In this case, there exists an inverse transformation S^{-1} with components T^i_j such that $S^i_k T^k_j = \delta^i_j$. It is evident that substitutions (2) form a local group of transformations, and equations for the field Ψ should be invariant with respect to this gauge symmetry group. As is well known, the matrix Ψ can be reduced by a gauge transformation to the canonical or Jordan form defined by the characteristic equation

$$|\Psi^i_j - \lambda \delta^i_j| = 0.$$

The Einstein general covariance is the other deep guiding principle that we have at our disposal. In this framework, the wave equation for Ψ is, in fact, defined uniquely.

The most direct and simplest way to derive a gauge-invariant wave equation is to give a correct definition of the gauge-covariant derivative. Denote it by D_i and let

$$D_i \Psi^j_k = \partial_i \Psi^j_k + G^j_{il} \Psi^l_k - \Psi^j_l G^l_{ik}$$

$$\bar{D}_i \bar{\Psi}^j_k = \partial_i \bar{\Psi}^j_k + \bar{G}^j_{il} \bar{\Psi}^l_k - \bar{\Psi}^j_l \bar{G}^l_{ik}$$

where G^j_{ik} and $\bar{G}^j_{ik}$ are gauge potentials connected with wave fields Ψ and $\bar{\Psi}$, respectively.

For brevity, we will use the matrix notation

$$\Psi = (\Psi^i_j) \qquad G_i = (G^j_{ik}) \qquad S = (S^i_j) \qquad S^{-1} = (T^i_j) \qquad \mathrm{Tr}(\Psi) = \Psi^i_i$$

in which

$$D_i \Psi = \partial_i \Psi + [G_i, \Psi] \qquad \bar{G}_i \bar{\Psi} = \partial_i \bar{\Psi} + [\bar{G}_i, \bar{\Psi}] \qquad \bar{\Psi} = S \Psi S^{-1}. \tag{4}$$

From the condition

$$\bar{D}_i \bar{\Psi} = S(D_i \Psi) S^{-1} \tag{5}$$

and (4) one derives the law of transformation of the gauge potential

$$\bar{G}_i = S G_i S^{-1} + S \partial_i S^{-1}. \tag{6}$$

To derive a general covariant Lagrangian of first order for Ψ, it is natural to take $D_i \Psi^j_k$ to be a tensor. Of course, the second gauge-covariant derivative should not be a tensor, but if we deal with the general covariant Lagrangian of first order, then by varying it we will obtain a combination of gauge-covariant derivatives such that equation of second order will be a tensor equation.

Let

$$\tilde{x}^i = \tilde{x}^i(x^0, x^1, x^2, x^3) \qquad x^i = x^i(\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$$

be the coordinate transformations. Put

$$X = \left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \qquad X^{-1} = \left(\frac{\partial x^i}{\partial \tilde{x}^j}\right)$$

then the gauge potential transforms as follows:

$$\tilde{G}_i = X G_k X^{-1} \frac{\partial x^k}{\partial \tilde{x}^i} + X \frac{\partial}{\partial \tilde{x}^i} X^{-1}.$$

Since the tensor field Ψ transforms under gauge and coordinate transformations by the law $\bar{\Psi} = S\Psi S^{-1}$, $\tilde{\Psi} = X\Psi X^{-1}$, the traces $\Psi^i_i = \mathrm{Tr}(\Psi)$ and $\Psi^i_j \Psi^j_i = \mathrm{Tr}(\Psi\,\Psi)$ are evidently invariants of the gauge group and scalars with respect to the general coordinate transformations. It is known from the theory of linear operators that there also exist other invariants, but in what follows we will only use these simplest ones.

To obtain an expression for the tensor of strength of the gauge field, consider the commutator of covariant derivatives $[D_i,\ D_j]$. From (4) it follows that

$$[D_i,\ D_j]\Psi = [H_{ij},\ \Psi] \tag{7}$$

where

$$H_{ij} = \partial_i G_j - \partial_j G_i + [G_i,\ G_j] \tag{8}$$

is the strength tensor of the gauge field with the following properties:

$$\bar{H}_{ij} = S H_{ij} S^{-1} \qquad [D_i,\ D_j] H_{kl} = [H_{ij},\ H_{kl}]. \tag{9}$$

From (8) it follows that H^k_{ijl} is a tensor of the type (1, 3) with respect to the general coordinate transformation.

3. Gauge-invariant equations

Now we have all that is required to derive the simplest general covariant and gauge-invariant equations for fields G_i and Ψ. In what follows g_{ij} are the Einstein gravitational potentials, g^{ij} are components of the tensor inverse to g_{ij}, $g_{il} g^{jl} = \delta^i_j$. As is known, the determinant $|g_{ij}| \neq 0$, actually allows one to obtain, for the tensor field g_{ij}, the equations invariant under the general coordinate transformations. By analogy, let us consider the case when the determinant $|\Psi^i{}_j| \neq 0$. Under this condition the field Ψ is inverse and the nonlinear gauge-invariant equations can be suggested. This means that we consider the case of nonlinear continuum mechanics with internal degrees of freedom. The linear case can be considered then as an approximation.

Thus, the gauge-invariant and general covariant Lagrangian of first order has the form

$$L = -\tfrac{1}{2} a\, \mathrm{Tr}(D_i \Psi D^i \Psi^{-1}) - \tfrac{1}{4} b\, \mathrm{Tr}(H_{ij} H^{ij}) \tag{10}$$

where a and b are constants,

$$D^i = g^{ij} D_j \qquad \text{and} \qquad H^{ij} = g^{ik} g^{jl} H_{kl}.$$

The gauge potential has the dimension of cm^{-1}, Ψ is dimensionless. Taking into account that

$$\delta\Psi = -\Psi(\delta\Psi^{-1})\Psi$$

and varying (10) with respect to Ψ and G_i, we obtain the following equations of second order for the basic fields:

$$D_i\big(\sqrt{|g|}\Psi^{-1}D^i\Psi\big) = 0 \tag{11}$$

$$D_i\big(\sqrt{|g|}H^{ij}\big) = \sqrt{|g|}J^j \tag{12}$$

where $|g|$ is the absolute value of the determinant of the matrix (g_{ij}) and

$$J^i = [\Psi^{-1}, D^i\Psi].$$

According to (9),

$$D_iD_j\big(\sqrt{|g|}H^{ij}\big) = 0.$$

Hence, as a result of (12), the tensor current J has to satisfy the equation

$$D_i\big(\sqrt{|g|}J^i\big) = 0.$$

Since this is really so, the system of equations (11) and (12) is compatible.

Varying the Lagrangian (10) with respect to g_{ij}, we obtain the gauge-invariant metric tensor of energy–momentum of oriented media

$$T_{ij} = a\,\mathrm{Tr}(D_i\Psi D_j\Psi^{-1}) + b\,\mathrm{Tr}(H_{ik}H^k_j) + g_{ij}L$$

which satisfies the equation

$$T^{ij}{}_{;i} = 0. \tag{13}$$

The semicolon denotes the covariant derivative with respect to the Levi-Civita connection belonging to the field g_{ij}

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \tfrac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}).$$

When deriving (13), besides equations (11) and (12), one should use the standard relations [12] for the Christoffel symbols $\{^{\,i}_{jk}\}$ and the identity

$$D_iH_{jk} + D_jH_{ki} + D_kH_{ij} = 0$$

which can easily be obtained with the help of relation (7). From (13) and the gauge invariance of the metric tensor of energy–momentum it follows that the Einstein equations

$$R_{ij} - \tfrac{1}{2}g_{ij}R = \frac{G}{c^3}T_{ij}$$

derived from the Lagrangian $L_f = L_g + L$, where $L_g = (c^3/G)R$ is the Einstein–Hilbert Lagrangian, will be compatible. Thus, it is shown that the continuum gravitates.

In the linear approximation $\Psi^i_j = \delta^i_j + M^i_j$, we have from (11) the following equation for the matrix $M = (M^i_j)$:

$$D_i\big(\sqrt{|g|}D^iM\big) = 0. \tag{14}$$

We will say that the continuum admits elastic deformations if the vector field V^i exists such that

$$D_iV^j = 0. \tag{15}$$

If equation (15) has a non-trivial solution, then from (14) and (15) it follows that the vector field $U^i = M^i_k V^k$ obeys the equation

$$D_i\big(\sqrt{|g|}D^i U^j\big) = 0 \tag{16}$$

that can be considered as a general covariant and gauge-covariant analogue of the known equation of elastic deformations of media.

It is worthwhile to pay attention to some special cases connected with the gauge group. First of all, the consideration of rigid directors or generalized Cosserat continua means the reduction of the gauge group defined by the condition

$$S^i_k S^j_l g_{ij} = g_{kl}.$$

Secondly, for a given vector field u^i a reduction of the gauge group is given by the relation

$$S^i_j u^j = u^i.$$

In the absence of a gravitational field we set $g_{ij} = \mathrm{diag}(1,\ -1,\ -1,\ -1)$. Finally, consider a gauge-invariant state in question. A state (Ψ, H_{ij}) is said to be a singlet if it is invariant under all the symmetry transformations. In our case a singlet state is given by the equations

$$\Psi = S\Psi S^{-1} \qquad H_{ij} = SH_{ij}S^{-1}$$

to be satisfied at any S. The first equation has the solution

$$\Psi = \mathrm{e}^{\alpha} E \qquad \Psi^{-1} = \mathrm{e}^{-\alpha} E$$

where α is a scalar field. In this case all directions are interchangeable. If the gauge field obeys the equation $H_{ij} = SH_{ij}S^{-1}$, it also obeys the equation $H_{ij} = \frac{1}{4}F_{ij}\,E$, where $F_{ij} = \mathrm{Tr}\,H_{ij}$. Thus, a singlet state is represented by the scalar field and 2-form $F_{ij}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$. To derive the equation for the scalar field consider the following gauge-invariant quantity: $\triangle = |\Psi^i_j|$. If Ψ obeys equation (11), then taking the trace of both sides of this equation we obtain that the invariant $\triangle$ satisfies the equation

$$\partial_i\big(\sqrt{|g|}g^{ij}\partial_j \ln|\triangle|\big) = 0.$$

Thus, one can derive that α must satisfy the equation

$$\partial_i\big(\sqrt{|g|}g^{ij}\partial_j \alpha\big) = 0.$$

From (12) it follows that the bivector F_{ij} satisfies the equations

$$\partial_i\big(\sqrt{|g|}F^{ij}\big) = 0$$

which coincide formally with the free Maxwell equations. This analogy can be prolonged since from (8) it follows that $F_{ij} = \partial_i Q_j - \partial_j Q_i$, where $Q_i = \mathrm{tr}\,G_i = G^k_{ik}$. According to (6) and the differentiation rules for determinants, the transformation law for Q_i under gauge transformations has the form

$$\bar{Q}_i = Q_i - \partial_i \ln|D|$$

where $D = \det(S^i_j)$.

4. Conclusion

Let us outline the main principles of the gauge approach to the continua with internal degrees of freedom. In the proposed theory directors are not fundamental objects and are treated as solutions of the algebraic equation (1) at a given matrix Ψ which is a wave field corresponding to the continuum in question and defines all the important quantities of the oriented media. The field Ψ is deduced from equations (11) and (12). The gravitational interactions of oriented media are described by the Einstein equations with the energy–momentum tensor of continua given by equation (13). For the equations presented, the gauge invariance holds necessarily in the sense that if Ψ and G_i are solutions then

$$\bar{\Psi} = S\Psi S^{-1} \qquad \bar{G}_i = SG_i S^{-1} + S\partial_i S^{-1}$$

are solutions as well, where S is any non-degenerate tensor field of the type (1, 1).

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